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Nonequilibrium dynamics of a growing interface

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Abstract
A growing interface subject to noise is described by the Kardar–Parisi–Zhang equation or, equivalently, the noisy Burgers equation. In one dimension this equation is analysed by means of a weak-noise canonical phase-space approach applied to the associated Fokker–Planck equation. The growth morphology is characterized by a gas of nonlinear soliton modes with superimposed linear diffusive modes. We also discuss the ensuing scaling properties.

1. Introduction

Macroscopic phenomena far from equilibrium abound and include phenomena such as turbulence in fluids, interface and growth problems, chemical reactions, processes in glasses and amorphous systems, biological processes, and even aspects of economical and sociological structures.

In recent years much of the focus of modern statistical physics and soft condensed matter has shifted towards such systems. Drawing on the case of static and dynamic critical phenomena in and close to equilibrium, where scaling, critical exponents, and the concept of universality have so successfully served to organize our understanding and to provide a variety of calculational tools, a similar approach has been advanced towards the much larger class of nonequilibrium phenomena with the purpose of elucidating scaling properties and more generally the morphology or pattern formation in a driven state.

In this context the noisy Burgers equation or the equivalent Kardar–Parisi–Zhang (KPZ) equation, describing the nonequilibrium growth of a noise-driven interface, provide simple continuum models of an open driven nonlinear system exhibiting scaling and pattern formation.

In one dimension the KPZ equation has the form (in a co-moving frame) [1, 2]

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta. \tag{1}$$

Here h is the height field, ν a damping or viscosity characterizing the linear diffusive term, λ a coupling strength for the nonlinear mode-coupling or growth term, and η a Gaussian white noise, driving the system into a stationary state. The noise is correlated according to

$$\langle \eta(xt)\eta(00)\rangle = \Delta\delta(x)\delta(t) \tag{2}$$

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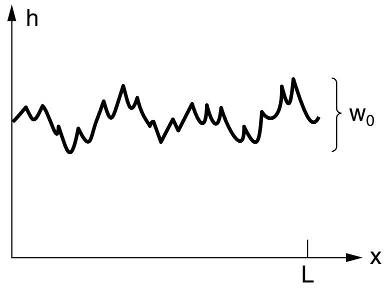

Figure 1. We depict the growth morphology of a growing interface in a system of size L. The saturation width in the noise-driven stationary state is denoted by w_0.

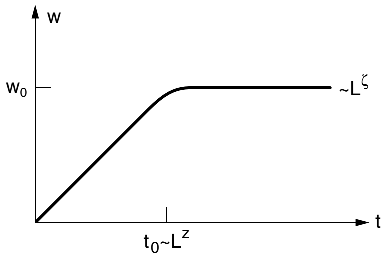

Figure 2. We depict the interface width $w(t)$ as a function of t. In the transient regime for $t \ll t_0 \sim L^z$, w grows according to $t^{\zeta/z}$. In the stationary regime attained for $t \gg t_0$, the width w saturates to the value $w_0 \sim L^\zeta$.

and characterized by the noise strength Δ. In figure 1 we have depicted a realization of a growing interface.

Assuming an initially flat interface, the width $w(t, L)$ grows in time approaching the saturation width w_0 in the stationary regime. The dynamical scaling hypothesis [3, 4] then asserts that

$$w(t, L) = L^\zeta F_{\rm w}(t/L^z) \tag{3}$$

where ζ is the roughness exponent characterizing the morphology of the interface and z the dynamic exponent describing the dynamical correlations. The scaling function $F_{\rm w}(u)$ (together with ζ and z) defines the scaling universality class and has the limits $F_{\rm w}(u) \to$ constant for $u \to \infty$ and $F_{\rm w}(u) \to u^{\zeta/z}$ for $u \to 0$. In figure 2 we have shown the time- and size-dependent width $w(t, L)$. Similarly, applying the dynamical scaling hypothesis to the height correlations in the stationary regime we have

$$\big\langle (h(xt) - \langle h(xt)\rangle)(h(00) - \langle h(00)\rangle)\big\rangle = x^{2\zeta} F_{\rm h}(t/x^z) \tag{4}$$

where the scaling function $F_{\rm h}$ obeys $F_{\rm h}(u) \to$ constant for $u \to \infty$ and $F_{\rm h}(u) \to u^{2\zeta/z}$ for $u \to 0$.

In what follows it turns out that the local slope field $u = \nabla h$ is the appropriate variable in terms of which to discuss the growth morphology of an interface. The height field $h = \int {\rm d}x\, u$ is then an integrated variable sampling the slope fluctuations. In figure 3 we have depicted a realization of the slope field. The dynamical scaling hypothesis for the slope field then reads

$$\langle u(xt)u(00)\rangle = x^{2\zeta-2} F_{\rm u}(t/x^z) \tag{5}$$

with limits $F_{\rm u}(u) \to$ constant for $u \to \infty$ and $F_{\rm u}(u) \to u^{(2\zeta-2)/z}$ for $u \to 0$.

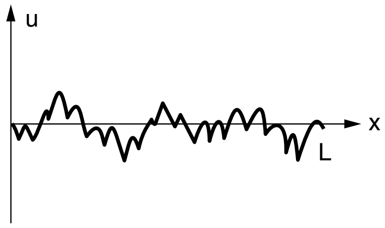

Figure 3. We depict the morphology of a growing interface in terms of the local slope. The slope field fluctuates about zero.

The stochastic dynamics of the slope field is then according to equation (1) governed by the noisy Burgers equation [5, 6]

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u + \lambda u \nabla u + \nabla \eta. \tag{6}$$

This equation in the noiseless case for $\eta = 0$ was originally proposed by Burgers [7] in order to model turbulence in fluids; note the similarity with the Navier–Stokes equation for $\lambda = -1$.

The substantial conceptual problems encountered in nonequilibrium physics are in many ways embodied in the KPZ–Burgers equations (1) and (6) describing the self-affine growth of an interface subject to annealed noise arising from fluctuations in the drive or in the environment, and these equations in one and higher dimensions have been the subject of intense scrutiny in recent years owing to their paradigmatic significance within a field theory of nonequilibrium systems [3, 4, 8–16].

Interestingly, the Burgers–KPZ equations are also encountered in a variety of other problems such as randomly stirred fluids, dissipative transport in a driven lattice gas, the propagation of flame fronts, the sine–Gordon equation, and magnetic flux lines in superconductors. Furthermore, by means of the Cole–Hopf transformation the Burgers–KPZ equations are also related to the problem of a directed polymer or a quantum particle in a random medium and thus to the theory of spin glasses.

In a series of papers, the one-dimensional case defined by (6) has been analysed in an attempt to uncover the physical mechanisms underlying the pattern formation and scaling behaviour. Emphasizing that the noise strength Δ constitutes the relevant nonperturbative parameter driving the system into a statistically stationary state, the method was initially based on a weak-noise saddle-point approximation to the Martin–Siggia–Rose functional formulation of the noisy Burgers equation [17–20]. This work was a continuation of earlier work based on the mapping of a solid-on-solid model onto a continuum spin model [21]. More recently the functional approach has been superseded by a *canonical phase-space method* deriving from the canonical structure of the Fokker–Planck equation associated with the Burgers equation [22–25]. In the present context we attempt to give a brief account of this approach with emphasis on the physical interpretation.

2. The noisy Burgers equation

The noisy Burgers equation (6) has the form of a conserved nonlinear Langevin equation, $\partial u/\partial t = -\nabla j$, with fluctuating current $j = -\nu \nabla u - (\lambda/2)u^2 - \eta$. The hydrodynamical origin of the Burgers equation, as reflected by the presence of the mode-coupling or convective term $\lambda u \nabla u$, implies that the Burgers equation is invariant under the Galilean transformation

$$x \to x - \lambda u_0 t \qquad \text{and} \qquad u \to u + u_0 \tag{7}$$

involving a shift of the slope field. Since the nonlinear coupling strength λ enters as a structural constant in the symmetry group, it is invariant under scaling and a simple Kadanoff-type block renormalization group argument in both space and time implies the dynamical scaling law

$$\zeta + z = 2 \tag{8}$$

providing a relationship between the roughness exponent and the dynamic exponent.

In the linear case for $\lambda = 0$ the Burgers equation takes the Edwards–Wilkinson (EW) form [26]

$$\partial u/\partial t = \nu \nabla^2 u + \nabla \eta \tag{9}$$

i.e., a linear diffusion equation driven by conserved noise. This equation is easily decomposed and analysed in terms of wavenumber modes $u_k = \int \mathrm{d}x\, u(x) \exp(-\mathrm{i}kx)$, $u_k^* = u_{-k}$. In the noiseless case for $\eta = 0$ the field u_k, governed by an overdamped oscillator equation, decays according to $u_k \propto \exp(-\nu k^2 t)$; $\tau_k = 1/\nu k^2$ setting the spectrum of relaxation times. For $\eta \neq 0$ the slope field u_k is driven into a noisy stationary state. Defining $u_{k\omega} = \int \mathrm{d}t\, u_k \exp(\mathrm{i}\omega t)$, the stationary correlations are given by

$$\langle u_{k\omega} u_{-k-\omega} \rangle = \frac{\Delta k^2}{\omega^2 + (\nu k^2)^2}. \tag{10}$$

We note that (10) in addition to the pole $\omega = \mathrm{i}\nu k^2$, corresponding to the decaying mode, also has a pole at $\omega = -\mathrm{i}\nu k^2$, characterizing a growing mode. The noise in driving the equation thus excites a growing diffusive mode and we have generally for a particular realization

$$u(xt) = (A\mathrm{e}^{-\nu k^2 t} + B\mathrm{e}^{+\nu k^2 t})\mathrm{e}^{\mathrm{i}kx}. \tag{11}$$

In the stationary state the correlations are time-reversal invariant, requiring both growing and decaying modes; this also follows from the Fokker–Planck analysis in section 3. From (10) we also obtain the static correlations $\langle u(x)u(0) \rangle = \Delta/2\nu\delta(x)$, showing that $u(x)$ is spatially uncorrelated. Moreover, the stationary distribution has the form

$$P_{\mathrm{st}}(u) \propto \exp\left[-\frac{\nu}{\Delta}\int \mathrm{d}x\, u(x)^2\right]. \tag{12}$$

Comparing (10) with the scaling form (5), we also infer the scaling exponents $\zeta = 1/2$ and $z = 2$, characteristic of diffusion and defining the EW universality class. Also, noting that the diffusive term in (9) can be derived from a free energy $F = (1/2)\int \mathrm{d}x\, u^2$, it follows that the EW equation describes the fluctuations in an equilibrium system with temperature $\Delta/2\nu$, i.e., $P_{\mathrm{st}} = \exp[-(2\nu/\Delta)F]$.

Summarizing, in the linear case the elementary excitations are decaying and growing nonpropagating diffusive modes. Since from (1) $\partial\langle h \rangle/\partial t = \nu \nabla^2 \langle h \rangle$, the average height $\langle h \rangle$ decays to zero (with respect to the co-moving frame) and the diffusive modes do not represent nonequilibrium growth but in fact characterize equilibrium fluctuations at the temperature $\Delta/2\nu$. Finally, the exponents $\zeta = 1/2$ and $z = 2$ for the EW universality class reflect the uncorrelated slope field, or the random walk of the integrated slope, i.e., the height h, and the diffusional character of the dynamics, respectively.

In the nonlinear case for $\lambda \neq 0$ the Galilean invariance (7) becomes operational and we can draw some simple conclusions concerning a growing interface. First, imposing a constant shift u_0 (7) implies that the tilted height field propagates with velocity λu_0 as shown in figure 4. Next, considering a step in the slope field with amplitudes u_1 and u_2 propagating

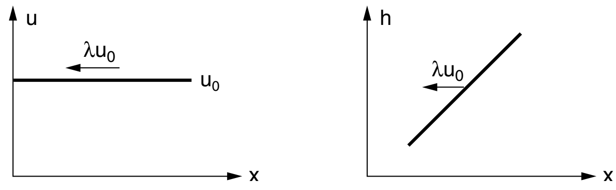

Figure 4. We show the slope and height fields for a constant shift u_0. The sloped height profile then moves with velocity λu_0.

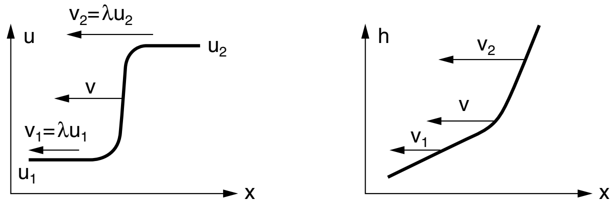

Figure 5. We show a step profile in the slope field with amplitudes u_1 and u_2. The step moves with mean velocity $v = (v_1 + v_2)/2$, where $v_1 = \lambda u_1$ and $v_2 = \lambda u_2$ are the velocities of the plateaus. The mode corresponds to the slope-dependent velocity of the height field subject to growth.

with velocities $v_1 = \lambda u_1$ and $v_2 = \lambda u_2$, it follows that the step itself propagates with mean velocity $v = (v_1 + v_2)/2$. For the height field this corresponds to the part with the largest slope moving with the largest velocity in accordance with the slope-dependent growth velocity in the KPZ equation (1), i.e., $\partial\langle h\rangle/\partial t = (\lambda/2)\langle(\nabla h)^2\rangle > 0$. These kinds of localized mode giving rise to growth are indeed supported by the Burgers equation which possesses a spectrum of soliton modes. The configuration is depicted in figure 5.

In the noiseless case for $\eta = 0$ the Burgers equation supports a right-hand single-soliton solution, in the static case of the kink-like form

$$u(x) = u_0 \tanh \frac{\lambda u_0}{2\nu}(x - x_0) \tag{13}$$

localized at x_0, with amplitude u_0, and width $2\nu/\lambda u_0$. Boosting the soliton to a finite velocity, this mode corresponds to the configuration depicted in figure 5. Denoting the right- and left-hand boundaries by u_+ and u_-, respectively, we also infer the soliton condition

$$u_+ + u_- = -2v/\lambda. \tag{14}$$

The relaxing growth morphology of the noiseless Burgers equation corresponding to the deterministic transient growth of the KPZ equation can be described by a gas of propagating right-hand solitons, connected by constant-slope ramp solutions, and with a spectrum of superimposed decaying linear modes. In the height field this morphology corresponds to downward cusps connected by parabolic segments with superimposed linear modes [1, 2, 27, 28].

In the noisy case the interface is driven into a stationary state, and anticipating the analysis in section 3, it turns out that the noise excites a left-hand soliton of the shape

$$u(x) = -u_0 \tanh \frac{\lambda u_0}{2\nu}(x - x_0). \tag{15}$$

This mode is a solution of the growing noiseless Burgers equation for $\nu \to -\nu$. This doubling of modes is equivalent to the linear case where the mode has the form (11). The stationary growth morphology can thus be described by a gas of right-hand and left-hand solitons matched by the soliton condition (14) with superposed linear modes.

The Fokker–Planck equation for the Burgers equation has the form

$$\Delta \frac{\partial P}{\partial t} = HP \tag{16}$$

where P is the probability distribution and H a Hamiltonian (Liouvillean) driving the equation of the form

$$H = -\Delta \int \mathrm{d}x\, \frac{\delta}{\delta u}(\nu\, \nabla^2 u + \lambda u\, \nabla u) + \frac{1}{2}\Delta^2 \int \mathrm{d}x\, \mathrm{d}x'\, \nabla\, \nabla' \delta(x - x') \frac{\delta^2}{\delta u\, \delta u}. \tag{17}$$

Whereas (16) will be discussed in more detail in section 3, it follows easily that the stationary solution of (16) has the form in (12) [29], independent of λ, and we infer as in the linear case $\zeta = 1/2$. The scaling law (8) then implies the dynamic exponent $z = 3/2$.

Summarizing, in the Burgers case the elementary excitations are propagating right-hand and left-hand solitons with superimposed linear modes. The soliton propagation corresponds to nonequilibrium growth. Finally, the scaling exponents $\zeta = 1/2$ and $z = 3/2$ defining the Burgers/KPZ universality class correspond to the uncorrelated slope field or random walk of h and to soliton propagation, respectively.

3. The weak-noise approach

Apart from numerical modelling and analysis of other models falling in the same universality class, the standard analytical approaches to the noisy Burgers equation are (i) the dynamical renormalization group (DRG) method and (ii) the mode-coupling (MC) approach. The DRG method accesses the scaling regime and provides an epsilon expansion about $d = 2$. Above $d = 2$ the system exhibits a kinetic phase transition from a smooth phase with EW exponents to a rough phase controlled by a strong-coupling fixed point with largely unknown scaling exponents. For $d = 1$ the scaling is controlled by a strong-coupling fixed point and the DRG yields (fortuitously) the known scaling exponents. Unlike the success of the DRG in dynamical critical phenomena, the results obtained for the noisy Burgers equation are limited despite a substantial theoretical effort. The MC method, by neglecting (unrenormalized) vertex corrections, provides closed equations yielding scaling functions. However, the somewhat *ad hoc* nature of the MC approach makes it difficult to make contact with more systematic approaches.

The functional or the equivalent phase-space approach valid in the weak-noise limit, $\Delta \to 0$, replaces the stochastic Langevin-type Burgers equation (6) by coupled deterministic diffusion–advection-type mean-field equations:

$$\frac{\partial u}{\partial t} = \nu\, \nabla^2 u - \nabla^2 p + \lambda u\, \nabla u \tag{18}$$

$$\frac{\partial p}{\partial t} = -\nu\, \nabla^2 p + \lambda u\, \nabla p \tag{19}$$

for the slope $u(x, t)$ and a canonically conjugate noise field $p(x, t)$, replacing the stochastic noise $\eta(x, t)$. The field equations bear the same relation to the Fokker–Planck equation (16) as the classical equations of motion bear to the Schrödinger equation in the semi-classical WKB approximation.

To justify the weak-noise limit we recall the analogy with the WKB approximation in quantum mechanics which, owing to its nonperturbative character, captures features like bound states and tunnelling amplitudes, which are generally inaccessible to perturbation theory. Therefore, we anticipate that the present weak-noise approach to the Burgers equation also accounts correctly, at least in a qualitative sense, for the stochastic properties even at larger noise strength. However, there may be an upper threshold value beyond which the system may enter a new stochastic or kinetic phase. In the one-dimensional case discussed here, the scaling behaviour is controlled by a single strong-coupling fixed point which can be accessed by the present weak-noise approach. In two and higher dimensions a dynamic renormalization group analysis predicts a kinetic phase transition at a critical noise strength (or coupling strength) and the weak-noise approach presumably fails.

The equations (18) and (19) derive from a principle of least action characterized by an action $S(u' \to u'', t)$ for an orbit $u'(x) \to u''(x)$ traversed in time t:

$$S(u' \to u'', t) = \int_{0,u'}^{t,u''} \mathrm{d}t\, \mathrm{d}x \left(p\frac{\partial u}{\partial t} - \mathcal{H} \right) \tag{20}$$

with Hamiltonian density

$$\mathcal{H} = p\big(\nu \nabla^2 u + \lambda u \nabla u - \tfrac{1}{2} \nabla^2 p\big). \tag{21}$$

The action is of central importance in the present approach and serves as a weight function for the noise-driven nonequilibrium configurations in much the same manner as the energy E in the Boltzmann factor $\exp(-\beta E)$ for equilibrium systems, where β is the inverse temperature. The dynamical action in fact replaces the energy in the context of the dynamics of stochastic nonequilibrium systems governed by a generic Langevin equation driven by Gaussian white noise. The action provides a methodological approach and yields access to the time-dependent and stationary probability distributions:

$$P(u' \to u'', t) \propto \exp\left[-\frac{S(u' \to u'', t)}{\Delta} \right] \tag{22}$$

$$P_{\mathrm{st}}(u'') = \lim_{t\to\infty} P(u' \to u'', t) \tag{23}$$

and associated moments, e.g., the stationary slope correlations

$$\langle u(xt)u(00)\rangle = \int \prod \mathrm{d}u\, u''(x)u'(0)P(u' \to u'', t)P_{\mathrm{st}}(u'). \tag{24}$$

The contact with the Fokker–Planck equation (16) in the weak-noise limit is established by noting that (22) inserted in (16) to leading order yields the Hamilton–Jacobi equation $\partial S/\partial t + H(p, u) = 0$ with canonical momentum $p = \delta S/\delta u$, where the Hamiltonian $H = \int \mathrm{d}x\, \mathcal{H}$.

The canonical formulation yields the conserved energy E (following from time translation invariance), the conserved momentum Π (from space translation invariance), and the conserved area M (from the Burgers equation with conserved noise):

$$E = \int \mathrm{d}x\, \mathcal{H} \tag{25}$$

$$\Pi = \int \mathrm{d}x\, u \nabla p \tag{26}$$

$$M = \int \mathrm{d}x\, u. \tag{27}$$

The field equations (18) and (19) determine orbits in a canonical u–p phase space where the dynamical issue in determining S and thus P is to find an orbit from u' to u'' in time t, p being a slaved variable. Note that unlike in dynamical system theory we are not considering the asymptotic properties of a given orbit. In general, the orbits in phase space lie on the manifolds determined by the constants of motion E, Π, and M. Here the zero-energy manifold $E = 0$ plays a special role in defining the stationary state. For vanishing or periodic boundary conditions for the slope field, the zero-energy manifold is composed of the transient submanifolds

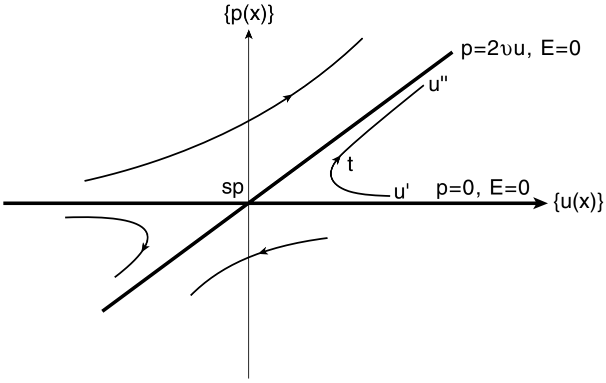

Figure 6. Generic behaviour of the orbits in u–p phase space. Heavy lines indicate the zero-energy manifold. The stationary saddle point (sp) is at the origin. The finite-time orbit from u' to u'' is attracted to the saddle point for $t \to \infty$.

$p = 0$ and the stationary submanifold $p = 2\nu u$. The zero-energy orbits on the $p = 0$ manifold correspond to solutions of the damped noiseless Burgers equation; the orbits on the $p = 2\nu u$ manifold are solutions of the undamped noiseless Burgers equation with negative damping, i.e., ν replaced by $-\nu$. In the solvable linear case of the noise-driven diffusion equation for $\lambda = 0$, i.e., the EW equation [26], a finite-energy orbit from $u' \to u''$ in time t migrates to the zero-energy manifold in the limit $t \to \infty$, yielding according to (20) and (23) the stationary distribution $P_{\mathrm{st}} \propto \exp(-(\nu/\Delta) \int \mathrm{d}x\, u^2)$. This distribution also holds in the Burgers case and is a generic result independent of λ [29]. Finally, in the long-time limit an orbit from $u' \to u''$ is attracted to the hyperbolic saddle point at the origin in phase space, implying ergodic behaviour in the stationary state. In figure 6 we have schematically depicted possible orbits in phase space.

In the linear case the field equations (18) and (19) couple p parametrically to u, i.e. p is slaved. In wavenumber space p_k is growing, whereas u_k driven by p_k is a linear superposition of damped and growing diffusive modes, supporting the expression (11). In the nonlinear case equations (18) and (19) admit nonlinear soliton or smoothed shock-wave solutions which are, in the static case, of the kink-like form in (13) and (15). Propagating solitons are subsequently generated by the Galilean boost (7) and we recover the soliton condition (14).

The right-hand soliton moves on the noiseless manifold $p = 0$ and is also a solution of the damped (stable) noiseless Burgers equation for $\eta = 0$. The noise-induced left-hand soliton is associated with the noisy manifold $p = 2\nu u$, and is a solution of the undamped (unstable) noiseless Burgers equation with ν replaced by $-\nu$. In addition the field equations also admit linear mode solutions superimposed as ripple modes on the solitons. The ripple modes are superpositions of both decaying and growing components reflecting the noiseless and noisy manifolds $p = 0$ and $2\nu u$, respectively. The soliton mode induces a propagating component with velocity λu in such a way that the right-hand soliton acts like a sink and the left-hand soliton as a source of linear modes [28]. This mechanism will be discussed heuristically in section 4. In the EW limit for $\lambda \to 0$ the ripple modes become the usual diffusive modes (growing and decaying) of the driven stationary diffusion equation. In figure 7 we have shown the right-hand and left-hand solitons.

The heuristic physical picture that emerges from our analysis, now supported by a weak-noise analysis of the Fokker–Planck equation, is that of a many-body formulation of the pattern formation of a growing interface in terms of a dilute gas of propagating solitons matched according to the soliton condition (14) with superimposed linear ripple modes.

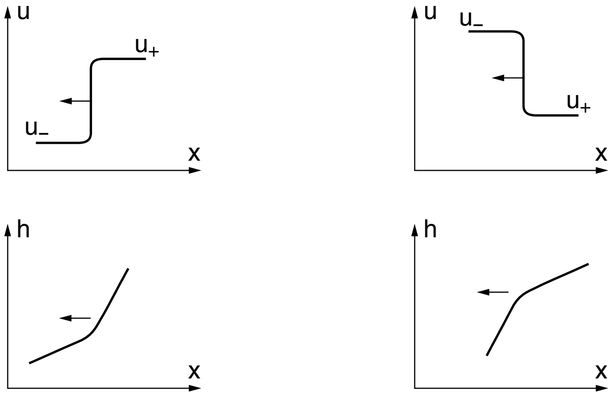

Figure 7. Slope field u and height profile h for the right-hand and left-hand moving kink solitons in the description of a growing interface.

4. A growing interface

As discussed above, a growing interface can be envisaged as a many-body system in the Landau quasi-particle sense, composed of matched right-hand and left-hand solitons with superimposed linear modes. Focusing on the solitons the right- and left-hand kinks are the fundamental growth modes corresponding to cusps in the height field; they are the quarks in the present formulation. They do not, however, satisfy periodic or vanishing boundary conditions in the slope field u; the nonvanishing boundary values u_+ and u_- in fact correspond to a deterministic current dissipated or generated at the soliton centres yielding permanent profile solutions. The simplest mode satisfying periodic boundary conditions is the two-soliton or pair soliton configuration obtained by matching a right-hand and a left-hand soliton boosted to the velocity $v = -\lambda u$. The two-soliton mode has amplitude $2u$ and size ℓ. By inspection, it is seen that the pair mode is an approximate solution to the field equations (18) and (19). The correction terms are of the type $u\,\nabla u$ and $u\,\nabla p$ and thus correspond to local perturbations from a region of size $\nu/\lambda|u|$ which is small in the low-viscosity limit $\nu \to 0$. We assume that the correction can be treated within a linear stability analysis and thus gives rise to a linear mode propagating between the right-hand and left-hand solitons. This property is borne out by a recent numerical analysis of the field equations [30]. As also shown in the numerical analysis, the pair mode forms a long-lived excitation or quasi-particle in the many-body description of a growing interface. Subject to periodic boundary conditions this mode corresponds to a simple growth situation. The propagation of the pair mode corresponds to the propagation of a step in the height field h. At each revolution of the pair mode, the interface grows by a uniform layer of thickness $2u\ell$. In figure 8 we have depicted the pair mode and the associated height profile h.

The soliton picture also allows us easily to understand in what sense the right-hand soliton acts like a drain and the left-hand soliton as a source with respect to perturbations. Considering two pair solitons superimposed on the right and left horizontal parts of the static solitons (13) and (15), it follows from (14) that for a right-hand soliton perturbations move toward the soliton centre and for a left-hand soliton perturbations move away from the soliton centre. This mechanism also follows from the linear analysis of ripple modes [28]. The mechanism is depicted in figure 9.

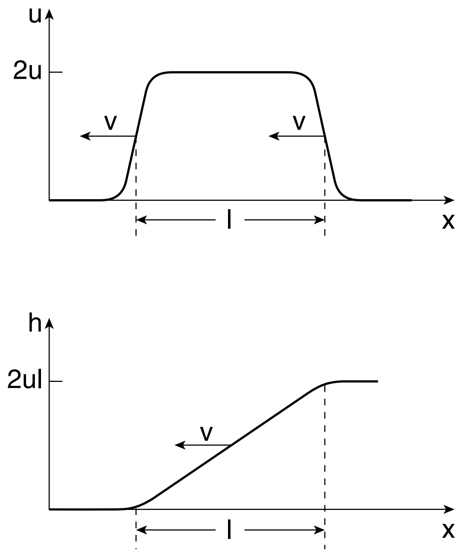

Figure 8. The slope field and the resulting height profile for a soliton pair configuration.

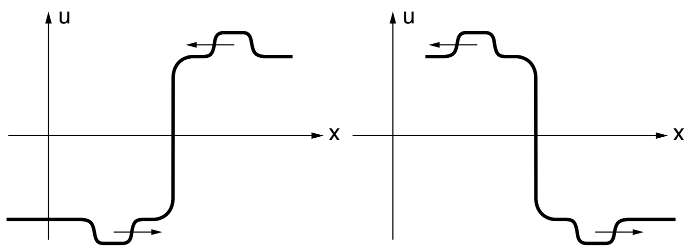

Figure 9. The source and drain mechanism for the right-hand and left-hand solitons. The perturbations attracted and repelled by the soliton centres are modelled by pair solitons.

Generally a growing interface, ignoring the superimposed linear ripple modes, can at a given time instant be represented by a gas of matched left-hand and right-hand solitons as depicted in figure 10 in the four-soliton case. A gas of pair solitons thus constitute a particular growth mode where the height profile between moving steps has horizontal segments.

5. Dynamics, statistics, and scaling

There are two levels of description: the stochastic Langevin level and the deterministic Fokker–Planck or equations-of-motion level. On the Fokker–Planck level yielding the canonical field equations (18) and (19), the growth of the interface is interpreted in terms of a gas of propagating solitons (and diffusive modes). The stochastic description on the Langevin level is then established in the weak-noise limit $\Delta \to 0$ by computing the action S associated with a particular dynamical mode and subsequently deducing the probability distribution according to (22), i.e., $P \propto \exp(-S/\Delta)$. This procedure is completely equivalent to the WKB limit of quantum mechanics. Here the wavefunction Ψ and thus the probabilistic interpretation is given by $\Psi \propto \exp(\mathrm{i}S/\hbar)$, where S is the action associated with the classical motion. Note that unlike in quantum mechanics there is no phase interference in the stochastic nonequilibrium case.

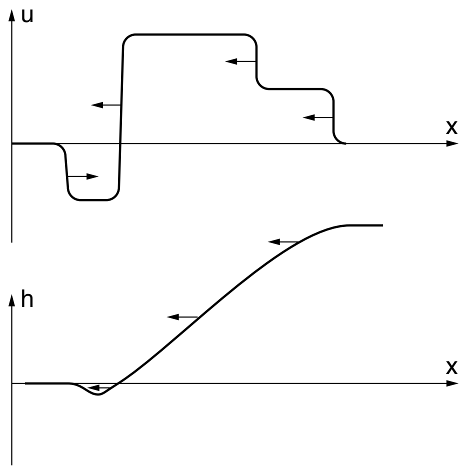

Figure 10. The four-soliton representation of the slope field and the height profile.

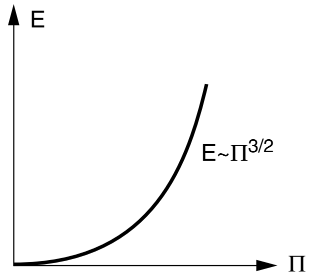

Figure 11. The dispersion law for the noise-induced left-hand soliton.

5.1. Dynamics

The canonical phase-space approach discussed in section 3 associates a formal dynamics with the soliton–diffusive mode gas representation of a growing interface. According to (25)–(27), the total energy, momentum, and area are conserved in the course of the dynamical evolution of an interface on the deterministic Fokker–Planck level as determined by the field equations (18) and (19). Since the symplectic structure via the Fokker–Planck equation is associated with the noisy case, only the noise-induced left-hand soliton and, similarly, the growing linear mode, carries dynamical attributes. By inspection, the static soliton (15) thus has the energy $E = -(16/3)\lambda\nu u_0^3$ and momentum $\Pi = 0$. Correspondingly, the moving pair soliton excitation depicted in figure 8 is endowed with the same energy but has momentum $\Pi = 4\nu u|u|$ pointing in the same direction as the velocity; it moreover conserves the area under propagation. Eliminating the amplitude dependence, the dynamical characteristics of the left-hand soliton are conveniently given by the soliton dispersion law

$$|E| = \frac{2}{3}\frac{\lambda}{\nu^{1/2}}\Pi^{3/2}. \tag{28}$$

We note the fractional power arising from the nonlinear character of the soliton; the dispersion law is shown in figure 11.

5.2. Statistics

We consider here as a simple application the statistics of the long-lived pair soliton shown in figure 8. It has size ℓ, amplitude $2u$, and propagates with velocity $v = -\lambda u$. During a revolution in a system of size L with periodic boundary conditions, the height field increases with a layer of thickness $2u\ell$. Since the system is traversed in time $t = L/v$ the integrated growth velocity is given by $2\lambda u^2\ell/L$, which for a single pair of fixed size vanishes in the thermodynamic limit. On the other hand, the local growth velocity $\mathrm{d}h/\mathrm{d}t$ is given by $2\lambda u^2 = (\lambda/2)(\nabla h)^2$ which is consistent with the averaged KPZ equation (1) in the stationary state.

The stochastic properties of the pair soliton growth mode are easily elucidated by noting that the action associated with the pair mode is given by $S = (4/3)\nu\lambda|u|^3 t$. Denoting the centre of mass of the pair mode by x, we have $u = v/\lambda = x/t\lambda$ and we obtain using (22) the transition probability

$$P(x,t) \propto \exp\left(-\frac{4}{3}\frac{\nu}{\Delta\lambda^2}\frac{x^3}{t^2}\right) \tag{29}$$

for the 'random walk' of independent pair solitons or steps in the height profile. Comparing (29) with the distribution for an ordinary random walk originating from the Langevin equation $\mathrm{d}x/\mathrm{d}t = \eta$, $\langle\eta\eta\rangle(t) = \Delta\delta(t)$, $P(x,t) \propto \exp(-x^2/2\Delta t)$, we observe that the growth mode performs anomalous diffusion. The distribution (29) also implies the soliton mean square displacement, assuming pairs of the same average size,

$$\langle x^2\rangle(t) \propto \left(\frac{\Delta\lambda^2}{\nu}\right)^{1/z} t^{2/z} \tag{30}$$

with dynamic exponent $z = 3/2$, identical to the dynamic exponent defining the KPZ universality class. This result should be contrasted with the mean square displacement $\langle x^2\rangle \propto \Delta t^{2/z}$, $z = 2$, for an ordinary random walk. The growth modes thus perform superdiffusion.

5.3. Scaling

Aspects of the scaling properties of the noisy Burgers equation are embodied in the scaling form (5) for the slope correlations. Here we give a set of heuristic arguments implying that the dynamic scaling exponent z can be inferred from the exponent in the soliton dispersion law (28); we refer the reader to [19, 31] for more details.

Within the weak-noise approximation the slope correlations are given by appropriate overlap integrals involving the soliton configuration [31]. However, noting that the canonical weak-noise formulation in general follows from a saddle-point approximation to the Martin–Siggia–Rose functional integral, we infer that the slope correlation can also be expressed as the time-ordered product [32] $\langle u(xt)u(00)\rangle \propto \langle 0|T\hat{u}(xt)\hat{u}(00)|0\rangle$. Here the 'quantum operators' $\hat{u}$ and $\hat{p}$ evolve according to the 'quantum Hamiltonian density' (21) and $|0\rangle$ denotes the zero-energy stationary state. Displacing the field from (x,t) to $(0,0)$, using the Hamiltonian and momentum operators, and inserting a complete set of intermediate quasi-particle momentum states $|\Pi\rangle$, we infer the spectral representation

$$\langle u(xt)u(00)\rangle \propto \int \mathrm{d}\Pi\, G(\Pi)\exp(Et - \mathrm{i}\Pi x). \tag{31}$$

Here $G(\Pi)$ is an effective form factor and E and Π the energy and momentum of the appropriate quasi-particle.

The scaling limit for large x and large t corresponds to the bottom of the quasi-particle spectrum and we note that only gapless excitations contribute. Assuming a general dispersion law with exponent β, $E \propto \Pi^{\beta}$, the dynamic exponent z is given by the exponent β for the quasi-particle dispersion law. In the linear EW case the gapless diffusive dispersion law $E \propto \Pi^2$ yields the dynamic exponent $z = 2$; in the Burgers–KPZ case the noise excites a new nonlinear gapless soliton mode with dispersion $E \propto \Pi^{3/2}$, yielding the exponent $z = 3/2$.

6. Summary and conclusions

We have here summarized recent work on the growth morphology and scaling behaviour of the noisy Burgers equation in one dimension. Using a canonical weak-noise approach to the associated Fokker–Planck equation, we have discussed the growth morphology in terms of a gas of nonlinear solitonic growth modes with superposed linear modes. We have, moreover, associated the dynamic scaling exponent with the soliton dispersion law.

So far the nonperturbative weak-noise approach has only been implemented in the one-dimensional case where the analysis is tractable. However, the weak-noise method is generally applicable to generic Langevin equations driven by white noise and it remains to be seen whether the approach also throws light on the higher-dimensional case.

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